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# ON THE SIZE OF SOME TRIGONOMETRIC POLYNOMIALS

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Let  $E_n(f)$  denote the best approximation of  $f \in C[0, 2\pi]$ , i.e.,

$$E_n(f) = \inf \|f - T_n\| ,$$

where  $\| \cdot \|$  is the supremum norm and  $T_n(x)$  are trigonometric polynomials of degree  $\leq n$ . It was Bernstein [3] who first showed the close connection between absolute convergence of the Fourier series of  $f(x) \in \text{Lip } \alpha$  and  $E_n(f)$ . In particular, he obtained the following theorem which shows his result being best possible.

Theorem A. For any given sequence  $\varepsilon_n \downarrow 0$  such that

$$\sum_{n=1}^{\infty} \varepsilon_n / \sqrt{n} = \infty ,$$

we can find an  $f \in C[0, 2\pi]$  whose Fourier series is not absolutely convergent at any point at all and yet satisfies the inequality  $E_n(f) \leq \varepsilon_n$ .

To prove this theorem, he invented the following lemma which may well deserve an independent interest.

Lemma A. For any given natural number  $N$ , we can find a trigonometric polynomial of the form

$$T_N(x) = \sum_{N/2 \leq n \leq N} \cos(nx + \rho_n)$$

such that uniformly in  $x$ ,

$$T_N(x) \ll \sqrt{N} .$$

Bernstein's original proof of this lemma is due to the theory of characters, while Bari [1] applied Kuzmin's lemma instead and Kahane<sup>[4]</sup> used Rudin-Shapiro theorem. Actually Bernstein considered the sum

$$(1) \quad S = \sum_{A \leq n \leq B} e(a(\frac{n^2}{N} + xn)), \quad (e(u) = \exp(2\pi i u))$$

where  $a > 0$ ,  $0 \leq x \leq 1$  and  $A, B, N \in \mathbb{N}$  are such that  $1 \leq A < B \leq N$ .

First we remark that if we apply Salem's lemma (Lemma 2 below) to  $S$ , then we obtain

$$(2) \quad S \ll (\sqrt{a} + 1/\sqrt{a})\sqrt{N},$$

which holds uniformly in  $x, A, B$ .

On the one hand, from a different stand point, there is a problem of finding the polynomials

$$(3) \quad P_N(z) = \sum_{n \leq N} c_{n,N} z^n,$$

with  $|c_{n,N}| = 1$  and  $|z| = 1$  such that

$$(4) \quad \sqrt{N} \ll P_N(z) \ll \sqrt{N},$$

for all  $z$ . Parseval's formula shows

$$(5) \quad \max_{|z|=1} |P_N(z)| \geq \sqrt{N}.$$

See e.g. Kahane[5] for recent results. The next example of (3) in the literature seems to be the following one due to Hardy and Littlewood[cf. 7, p.199]:

$$(6) \quad P_N(z) = \sum_{n \leq N} e(cn \log n + xn), \quad z = e(x)$$

which satisfies  $P_N(z) \ll \sqrt{N}$  uniformly in  $x$ . However, as far as I know, it seems open whether it satisfies  $P_N(z) \gg \sqrt{N}$  for all  $x$ . We notice that in their example the coefficients  $c_n = e(cn \log n)$  are independent of  $N$ .

The main purpose of this note is to show that the size of (3) may be sometimes smaller than  $\sqrt{N}$ . We shall show it effectively by constructing examples.

First we prove

Theorem 1. For any given  $N > 1$ , we can find a sequence  $c_{n,N} \in \mathbb{C}$  with  $|c_{n,N}| = 1$  ( $1 \leq n \leq N$ ) such that

$$(7) \quad \sum_{n \leq N} c_{n,N} z^n \ll N^{1/4},$$

for all  $z \in \mathbb{C}$  with  $|z| = 1$ , where  $\ll$  depends on  $z$ .

Proof. Consider the sum

$$(8) \quad S_N = \sum_{n \leq N} e(xn - 2s\sqrt{n}),$$

where  $0 < x \leq 1$  and  $s \geq 1$  will be suitably chosen (as a function of  $N$ ) later. If we put  $f(t) = xt - 2s\sqrt{t}$  ( $1 \leq t \leq N$ ), then

$$-s/\sqrt{t} < f'(t) \leq 1 - s/\sqrt{t} \leq 1 - s/\sqrt{N}.$$

Therefore, if  $4s^2 \leq t \leq N$ , then

$$|f'(t)| \leq 1 - s/\sqrt{N},$$

because then  $-(1 - s/\sqrt{N}) \leq -s/\sqrt{t}$ .

Now we shall apply the following known lemma due to van der Corput [cf.2].

Lemma 1. If  $f'(t)$  is monotone and satisfies

$$|f'(t)| \leq 1 - \varepsilon, \quad (0 < \varepsilon < 1)$$

throughout  $(a, b)$ , then

$$\sum_{a \leq n \leq b} e(f(n)) = \int_a^b e(f(t)) dt + O(1/\varepsilon),$$

where the constant implied by  $O$  is absolute.

If we insert  $\varepsilon = s/\sqrt{N}$  in the above lemma, then we obtain

$$\sum_{4s^2 \leq n \leq N} e(xn - 2s\sqrt{n}) = \int_{4s^2}^N e(xt - 2s\sqrt{t}) dt + O(\sqrt{N}/s).$$

Thus we have

$$(9) \quad S_N = \sum_{1 \leq n \leq 4s^2} e(xn - 2s\sqrt{n}) + \int_{4s^2}^N e(xt - 2s\sqrt{t}) dt + O(\sqrt{N}/s).$$

We appeal to the known lemma below to estimate the first sum in (9).

Lemma 2 (Salem[cf. 7, p.226]). If  $f''(t) > 0$  is monotone, then

$$\sum_{a \leq n \leq b} e(f(n)) = O\left(\max_{a \leq t \leq b} \frac{1}{\sqrt{f''(t)}}\right) + O\left(\int_a^b (\sqrt{f''(t)} + f''(t)) dt\right),$$

where the implied constants by  $O$ 's are absolute.

Now for  $f(t) = xt - 2s\sqrt{t}$  we have  $f''(t) = \frac{s}{2} t^{-3/2}$ . Hence by Lemma 2 we have

$$(10) \quad \sum_{1 \leq n \leq 4s^2} e(f(n)) = O\left(\max_{1 \leq t \leq 4s^2} \frac{1}{\sqrt{s}} t^{3/4}\right) + O\left(\int_1^{4s^2} \sqrt{st}^{-3/4} dt + \int_1^{4s^2} st^{-3/2} dt\right) = O(s) + O(s) + O(s) = O(s).$$

Next we shall estimate the integral

$$I_N = \int_{4s^2}^N e(xt - 2s\sqrt{t}) dt.$$

If we put  $t = u^2$ , then

$$\begin{aligned} I_N &= 2 \int_{2s}^{\sqrt{N}} u \cdot e(xu^2 - 2su) du \\ &= \frac{1}{2\pi i x} \int_{2s}^{\sqrt{N}} (e(xu^2 - 2su))' du + \frac{2s}{x} \int_{2s}^{\sqrt{N}} e(xu^2 - 2su) du \end{aligned}$$

$$= \frac{2s}{x} \int_{2s}^{\sqrt{N}} e(xu^2 - 2su) du + O(1/x).$$

Lemma 3 [cf. 6 & 7]. If  $f''(t) \geq r > 0$  throughout  $(a, b)$ , then

$$\int_a^b e(f(t)) dt \ll 1/\sqrt{r},$$

where  $\ll$  is absolute.

From this lemma we have

$$(11) \quad \int_{2s}^{\sqrt{N}} e(xu^2 - 2su) du = O(1/\sqrt{x}).$$

Thus we obtain from (9)-(11)

$$S_N = O(s) + O(sx^{-3/2}) + O(\sqrt{N}/s) + O(1/x).$$

Finally, by choosing  $s = \frac{1}{2} N^{1/4}$ , we get

$$S_N = O(N^{1/4}),$$

where the implied constant by  $O$  depends on  $x$ .  $\square$

If the coefficients  $c_{n,N}$  are independent of  $N$ , then the situation in general becomes more difficult and we then have the following result.

Theorem 2. For any given  $\varepsilon > 0$ , there exist a natural number  $N_0 = N_0(\varepsilon)$  and a sequence  $c_n = c_n(\varepsilon) \in \mathbb{C}$  with  $|c_n| = 1$  ( $1 \leq n \leq N$ ) such that for all  $N \geq N_0$  and  $z$  with  $|z| = 1$ ,

$$\sum_{n \leq N} c_n z^n \ll_{\varepsilon, z} N^{2/5 + \varepsilon}$$

Proof. We only indicate the outline of the proof since it is similar to that of Theorem 1. In this case we consider the sum

$$S_N = \sum_{n \leq N} e(xn - n^c/c),$$

where  $0 < x \leq 1$  and  $0 < c < 1$ . If we put  $f(t) = xt - t^c/c$  ( $2 \leq t \leq N$ ), then we have by Lemma 1

$$\sum_{2 \leq n \leq N} e(f(n)) = \int_2^N e(f(t)) dt + O(N^{1-c}).$$

Next we apply a known lemma [7, p.62] in order to estimate the above integral, say  $I(N)$ . Then after simple calculation, we have for

$$N \geq 2(2/x)^{1/(1-c)}$$

$$I(N) - I(N/2) = O(N^{1-3c/5}),$$

where  $O$  depends on  $c$  and  $x$ . Hence substituting in  $N$  successively

$$N/2, N/2^2, \dots$$

and adding them all, we get  $I(N) = O(N^{1-3c/5})$ .

Therefore we finally obtain

$$\begin{aligned} S_N &= O(N^{1-3c/5}) + O(N^{1-c}) = O(N^{1-3c/5}) \\ &= O(N^{2/5 + \varepsilon}), \quad (c = 1 - 5\varepsilon/3). \end{aligned}$$

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